

# ON SOME NONLINEAR EVOLUTION EQUATION OF SECOND ORDER

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**ABSTRACT.** Here we study the abstract nonlinear differential equation of second order that in special case is the equation of the type of equation of traffic flow. We prove the solvability theorem for the posed problem under the appropriate conditions and also investigate the behaviour of the solution.

## 1. INTRODUCTION

In this article we study the following nonlinear evolution equation

$$(1.1) \quad x_{tt} + A \circ F(x) = g\left(x, A^{-\frac{1}{2}}x_t\right), \quad t \in (0, T), \quad 0 < T < \infty$$

under the initial conditions

$$(1.2) \quad x(0) = x_0, \quad x_t(0) = x_1$$

here  $A$  is a linear operator in a real Hilbert space  $H$ ,  $F : X \rightarrow X^*$  and  $g : D(g) \subseteq H \times H \rightarrow H$  are a nonlinear operators,  $X$  is a real Banach space. For example, operator  $A$  denotes  $-\Delta$  with Dirichlet boundary conditions (such as homogeneous or periodic) and  $f, g$  are functions such as above, that in the one space dimension case, we can formulate in the form

$$(1.3) \quad u_{tt} - (f(u)u_x)_x = g(u), \quad (t, x) \in R_+ \times (0, l), \quad l > 0,$$

$$(1.4) \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad u(t, 0) = u(t, l),$$

where  $u_0(x)$ ,  $u_1(x)$  are known functions,  $f(\cdot), g(\cdot) : R \rightarrow R$  are a continuous functions and  $l > 0$  is a number. The equation of type (1.3) describe mathematical model of the problem from theory of the flow in networks as is affirmed in articles [1], [3], [4], [5] (e. g. Aw-Rascle equations, Antman–Cosserat model, etc.). As in the survey [3] is noted such a study can find application in accelerating missiles and space crafts, components of high-speed machinery, manipulator arm, microelectronic mechanical structures, components of bridges and other structural elements. Balance laws are hyperbolic partial differential equations that are commonly used to express the fundamental dynamics of open conservative systems (e.g. [4]). As the survey [3] possess of the sufficiently exact explanations of the significance of equations of such type therefore we not stop on this theme.

This article is organized as follows. In the section 2 we study the solvability of the nonlinear equation of second order in the Banach spaces, for which we found the sufficient conditions and proved the existence theorem. In the section 3 we investigate the global behaviour of solutions of the posed problem.

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## 2. SOLVABILITY OF PROBLEM (1.1) - (1.2)

Let  $A$  is a symmetric linear operator densely defined in a real Hilbert space  $H$  and positive,  $A$  has a self-adjoint extension, moreover there is linear operator  $B$  defined in  $H$  such that  $A \equiv B^* \circ B$ , here  $f : R \rightarrow R$  is continuous as function,  $X$  is a real reflexive Banach space and  $X \subset H$ ,  $g : D(g) \subseteq H \times H \rightarrow H$ , where  $g : R^2 \rightarrow R$  is a continuous as function and  $x : [0, T) \rightarrow X$  is an unknown function. Let  $F(r)$  as a function is defined as  $F(r) = \int_0^r f(s) ds$ . Let the inequation  $\|x\|_H \leq \|Bx\|_H$  is valid for any  $x \in D(B)$ . We denote by  $V$ ,  $W$  and by  $Y$  the spaces defined as  $V \equiv \{y \in H \mid By \in H\}$ ,  $W = \{x \in H \mid Ax \in H\}$  and as  $Y \equiv \{x \in X \mid Ax \in X\}$ , respectively, for which inclusions  $W \subset V \subset H$  are compact and  $Y \subset W$ .

Let  $H$  is the real separable Hilbert space,  $X$  is the reflexive Banach space and  $X \subset H \subset X^*$ ;  $V$  is the previously defined space. It is clear that  $W \subset V \subset H \subset V^* \subset W^*$  are a framed spaces by  $H$ , these inclusions are compact and  $X \subset V^* \subset W^*$  then one can define the framed spaces  $Y \subset V \subset H \subset V^* \subset Y^*$  then  $X \subset V^* \subset Y^*$  are compact, with use the property of the operator  $A$ . Assume that operator  $A$  such as  $A : V_B \rightarrow V_B^*$  and  $A : X^* \rightarrow Y^*$ . Consequently, we get  $A \circ F : X \rightarrow Y^*$  and  $A \circ F \circ A : Y \rightarrow Y^*$ . Moreover we assume that  $[X^*, Y]_{\frac{1}{2}} \subseteq V$ .

Since operator  $A$  is invertible, here one can set the function  $y(t) = A^{-1}x(t)$  for any  $t \in (0, T)$ , in the other words one can assume the denotation  $x(t) = Ay(t)$ .

We will interpret the solution of the problem (1.1) - (1.2) by the following manner.

**Definition 1.** A function  $x : (0, T) \rightarrow X$ ,  $x \in C^0(0, T; X) \cap C^1(0, T; V^*) \cap C^2(0, T; Y^*)$ ,  $x = Ay$ , is called a weak solution of problem (1.1) - (1.2) if  $x$  a. e.  $t \in (0, T)$  satisfies the following equation

$$(2.1) \quad \frac{d^2}{dt^2} \langle x, z \rangle + \langle A \circ F(x), z \rangle = \langle g(x, By_t), z \rangle$$

for any  $z \in Y$  and the initial conditions (1.2) (here and farther the expression  $\langle \cdot, \cdot \rangle$  denotes the dual form for the pair: the Banach space and his dual).

Consider the following conditions

(i) Let  $A : W \subset H \rightarrow H$  is the selfadjoint and positive operator, moreover  $A : V \rightarrow V^*$ ,  $A : X^* \rightarrow Y^*$ , there exists an linear operator  $B : V \rightarrow H$  that satisfies the equation  $Ax \equiv (B^* \circ B)x$  for any  $x \in D(A)$  and  $\|x\|_H \leq \|Bx\|_H = \|x\|_V$ .

(ii) Let  $F : X \rightarrow X^*$  is the continuously differentiable and monotone operator with the potential  $\Phi$  that is the functional defined on  $X$  (his Frechet derivative is the operator  $F$ ). Moreover for any  $x \in X$  the following inequalities hold

$$\|F(x)\|_{X^*} \leq a_0 \|x\|_X^{p-1} + a_1 \|x\|_H; \quad \langle F(x), x \rangle \geq b_0 \|x\|_X^p + b_1 \|x\|_H^2,$$

where  $a_0, b_0 > 0$ ,  $a_1, b_1 \geq 0$ ,  $p > 2$  are numbers.

(iii) Assume  $g : H \times V \rightarrow H$  is a continuous operator that satisfies the condition

$$|\langle g(x, y) - g(x_1, y_1), z \rangle| \leq g_1 |\langle x - x_1, z \rangle| + g_2 |\langle y - y_1, z \rangle|,$$

for any  $(x, y), (x_1, y_1) \in H \times H$ ,  $z \in H$  and consequently for any  $(x, y) \in H \times H$  the inequation

$$\|g(x, y)\|_H \leq g_1 \|x\|_H + g_2 \|y\|_H + g_0, \quad g_0 \geq \|g(0, 0)\|_H$$

holds, where  $g_0$  is a number.

In the beginning for the investigation of the posed problem we set the following expression in order to obtain of the a priori estimations

$$\langle x_{tt}, y_t \rangle + \langle A \circ F(x), y_t \rangle = \langle g(x, By_t), y_t \rangle$$

here element  $y$  is defined as the solution of the equation  $Ay(t) = x(t)$ , i.e.  $y(t) = A^{-1}x(t)$  for any  $t \in (0, T)$  as was already mentioned above.

Hence follow

$$\langle By_{tt}, By_t \rangle + \langle F(x), x_t \rangle = \langle g(x, By_t), y_t \rangle,$$

or

$$(2.2) \quad \frac{1}{2} \frac{d}{dt} \|By_t\|_H^2 + \frac{d}{dt} \Phi(x) = \langle g(x, By_t), y_t \rangle,$$

where  $\Phi(x)$  is the functional that defined as  $\Phi(x) = \int_0^1 \langle F(sx), x \rangle ds$  (see, [6]).

Then using condition (iii) on  $g(x, By_t)$  in (2.2) one can obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|By_t\|_H^2 + \frac{d}{dt} \Phi(x) &\leq \|g(x, By_t)\|_H^2 + \|y_t\|_H^2 \leq \\ 2 \left( g_1^2 \|x\|_H^2 + g_2^2 \|By_t\|_H^2 + g_0^2 \right) + \|y_t\|_H^2 &\leq \tilde{C} \left( \|x\|_H^2 + \frac{1}{2} \|By_t\|_H^2 + g_0^2 \right) \end{aligned}$$

here one can use the estimation  $\|x\|_H^2 \leq \tilde{c}(\Phi(x) + 1)$  (if  $b_1 > 0$  then  $\|x\|_H^2 \leq \tilde{c}\Phi(x)$ ) as  $2 < p$  by virtue of the condition (ii). Consequently we get to the Cauchy problem for the inequation

$$(2.3) \quad \frac{d}{dt} \left( \frac{1}{2} \|By_t\|_H^2(t) + \Phi(x(t)) \right) \leq C_0 \left( \frac{1}{2} \|By_t\|_H^2(t) + \Phi(x(t)) \right) + C_1$$

with the initial conditions

$$(2.4) \quad x(t)|_{t=0} = x_0; \quad y_t(t)|_{t=0} = A^{-1}x_t|_{t=0} = A^{-1}x_1$$

where  $C_j \geq 0$  are constants independent of  $x(t)$ . From here follows

$$\frac{1}{2} \|By_t\|_H^2(t) + \Phi(x(t)) \leq e^{tC_0} \left[ \|By_1\|_2^2 + 2\Phi(x_0) \right] + \frac{C_1}{C_0} (e^{tC_0} - 1).$$

This give to we the following estimations for every  $T \in (0, \infty)$

$$(2.5) \quad \|By_t\|_H^2(t) \leq C(x_0, x_1) e^{C_0 T}, \quad \Phi(x(t)) \leq C(x_0, x_1) e^{C_0 T},$$

for a. e.  $t \in (0, T)$ , i.e.  $y = A^{-1}x$  is contained in the bounded subset of the space  $y \in C^1(0, T; V) \cap C^0(0, T; Y)$ , consequently we obtain that if the weak solution  $x(t)$  exists then it belong to a bounded subset of the space  $C^0(0, T; X) \cap C^1(0, T; V^*)$ .

Hence one can wait, that the following inclusion

$$y \in C^2(0, T; X^* \cap H) \cap C^1(0, T; X^* \cap V) \cap C^0(0, T; Y)$$

holds by virtue of (2.1) in the assumption that  $x = Ay$  is a solution of the posed problem in the sense of Definition 1.

In order to prove of the solvability theorem we will use the Galerkin approach. Let the system  $\{y^k\}_{k=1}^\infty \subset Y$  be total in  $Y$  such that it is complete in the spaces  $Y, V$ , and also in the spaces  $X, H$ . We will seek out of the approximative solutions  $y_m(t)$ , consequently and  $x_m(t)$ , in the form

$$x_m(t) \equiv Ay_m(t) = \sum_{k=1}^m c_i(t) Ay^k \text{ or } x_m(t) \in \text{span} \{y^1, \dots, y^m\}$$

as the solutions of the problem locally with respect to  $t$ , where  $c_i(t)$  are as the unknown functions that will be defined as solutions of the following Cauchy problem for system of ODE

$$\begin{aligned} \frac{d^2}{dt^2} \langle x_m, y^j \rangle + \langle F(x_m), Ay^j \rangle &= \langle g(x_m, By_{mt}), y^j \rangle, \quad j = 1, 2, \dots, m \\ x_m(0) &= x_{0m}, \quad x_{tm}(0) = x_{1m}, \end{aligned}$$

where  $x_{0m}$  and  $x_{1m}$  are contained in  $\text{span} \{y^1, \dots, y^m\}$ ,  $m = 1, 2, \dots$ , moreover

$$x_{0m} \longrightarrow x_0 \quad \text{in} \quad [X, Y]_{\frac{1}{2}} \subseteq V; \quad x_{1m} \longrightarrow x_1 \quad \text{in} \quad X, \quad m \nearrow \infty.$$

Thus we obtain the following problem

$$(2.6) \quad \frac{d^2}{dt^2} \langle x_m, y^j \rangle + \langle F(x_m), Ay^j \rangle = \langle g(x_m, By_{mt}), y^j \rangle, \quad j = 1, 2, \dots, m$$

$$\langle x_m(t), y^j \rangle|_{t=0} = \langle x_{0m}, y^j \rangle, \quad \frac{d}{dt} \langle x_m(t), y^j \rangle|_{t=0} = \langle x_{1m}, y^j \rangle$$

that solvable by virtue of estimates (2.5) on  $(0, T)$  for any  $m = 1, 2, \dots$ ,  $j = 1, 2, \dots$  and  $T > 0$ . Hence we set

$$(2.7) \quad \frac{d^2}{dt^2} \langle x_m, z \rangle + \langle F(x_m), Az \rangle = \langle g(x_m, By_{mt}), z \rangle$$

for any  $z \in Y$  and  $m = 1, 2, \dots$

Consequently with use of the known procedure ([7], [8], [10]) we obtain,  $y_{mt} \in C^0(0, T; V)$ ,  $y_m \in C^0(0, T; Y)$  and  $x_m \in C^0(0, T; X)$ ,  $x_{mt} \in C^0(0, T; V^*)$ , moreover they are contained in the bounded subset of these spaces for any  $m = 1, 2, \dots$ . Hence from (2.5) we get

$$x_{mtt} \in C^0(0, T; Y^*) \quad \text{or} \quad x_m \in C^2(0, T; Y^*), \quad (V^* \subset Y^*).$$

Thus we obtain, that the sequence  $\{x_m\}_{m=1}^\infty$  of the approximated solutions of the problem is contained in a bounded subset of the space

$$C^0(0, T; X) \cap C^1(0, T; V^*) \cap C^2(0, T; Y^*)$$

or  $\{x_m\}_{m=1}^\infty$  such that for a. e.  $t \in (0, T)$  takes place the following inclusions  $\{y_m(t)\}_{m=1}^\infty \subset Y \subset X \subset H$ ,  $\{y_{mt}(t)\}_{m=1}^\infty \subset V$ ,  $\{y_{mtt}(t)\}_{m=1}^\infty \subset X^*$ . So we have

$$\{y_m(t)\}_{m=1}^\infty \subset C^0(0, T; Y) \cap C^1(0, T; V) \cap C^2(0, T; X^*)$$

therefore  $\{y_m(t)\}_{m=1}^\infty$  possess a precompact subsequence in  $C^1\left(0, T; [X^*, Y]_{\frac{1}{2}}\right)$  and in  $C^1(0, T; V)$ , as  $[X^*, Y]_{\frac{1}{2}} \subseteq V$  by virtue of conditions on  $X$  and  $A$  (by virtue of well known results, see, e. g. [2], [9] etc.). From here follows  $y_m(t) \longrightarrow y(t)$  in  $C^1(0, T; V)$  for  $m \nearrow \infty$  (Here and hereafter in order to abate the number of index we don't changing of indexes of subsequences). Then the sequence  $\{F(Ay_m(t))\}_{m=1}^\infty \subset X^*$  and bounded for a. e.  $t \in (0, T)$ ; the sequence

$$\{g(x_m(t), x_{mt}(t))\}_{m=1}^\infty \equiv \{g(Ay_m(t), By_{mt}(t))\}_{m=1}^\infty \subset H$$

and bounded for a. e.  $t \in (0, T)$  also, by virtue of the condition (iii). Indeed, for any  $m$  the estimation

$$\|g(Ay_m, By_{mt})\|_H(t) \leq \|Ay_m(t)\|_H + \|By_{mt}(t)\|_H + \|g(0, 0)\|_H$$

holds and therefore  $\{g(Ay_m(t), By_{mt}(t))\}_{m=1}^\infty$  is contained in a bounded subset of  $H$  for a. e.  $t \in (0, T)$ . Consequently  $\{F(Ay_m)\}_{m=1}^\infty$  and  $\{g(Ay_m(t), By_{mt}(t))\}_{m=1}^\infty$  have an weakly converging subsequences to  $\eta(t)$  and  $\theta(t)$  in  $X^*$  and  $H$ , respectively,

for a. e.  $t \in (0, T)$ . Hence one can pass to the limit in (2.7) with respect to  $m \nearrow \infty$ . Then we obtain the following equation

$$(2.8) \quad \frac{d^2}{dt^2} \langle x, z \rangle + \langle A\eta(t), z \rangle = \langle \theta(t), z \rangle.$$

So for us is remained to show the following: if the sequence  $\{x_m(t)\}_{m=1}^\infty \equiv \{Ay_m(t)\}_{m=1}^\infty$  is weakly converging to  $x(t) = Ay(t)$  then  $\eta(t) = F(x(t))$  and  $\theta(t) = g(x(t), By_t(t))$ . In order to show these equations are fulfilled we will use the monotonicity of  $F$  and the condition (iii).

We start to show  $\theta(t) = g((t), By_t(t))$  as  $x \in X \subset H$  and  $y_t \in V$ ,  $By_t \in H$  therefore  $g(x, By_t)$  is defined for a. e.  $t \in (0, T)$ . Consequently one can consider of the expression

$$\langle g(Ay_m(t), By_{mt}(t)) - g(Ay(t), By_t(t)), \hat{y} \rangle$$

for any  $\hat{y} \in C^0(0, T; Y) \cap C^1(0, T; V)$ . So we set this expression and investigate this for any  $\hat{y} \in C^0(0, T; Y) \cap C^1(0, T; V)$  then we have

$$(2.9) \quad |\langle g(Ay_m(t), By_{mt}(t)) - g(Ay(t), By_t(t)), \hat{y} \rangle| \leq g_1 |\langle Ay_m(t) - Ay(t), \hat{y}(t) \rangle| + g_2 |\langle By_{mt}(t) - By_t(t), \hat{y}(t) \rangle|$$

that takes place by virtue of the condition (iii). Using here the weak convergences of  $Ay_m(t) \rightharpoonup Ay(t)$  and  $By_{mt}(t) \rightarrow By_t(t)$  and by passing to the limit in the inequatin (2.9) with respect to  $m : m \nearrow \infty$  we get

$$|\langle \theta(t) - g(Ay(t), By_t(t)), \hat{y} \rangle| \leq 0$$

for any  $\hat{y} \in C^0(0, T; H)$ . Consequently the equation  $\theta(t) = g((t), By_t(t))$  holds, then the following equation is valid

$$\frac{d^2}{dt^2} \langle x, z \rangle + \langle A\eta(t), z \rangle = \langle g(x(t), By_t(t)), z \rangle$$

for any  $z \in Y$ , as  $\{y^k\}_{k=1}^\infty$  is complete in  $Y$  that display fulfilling of equation

$$(2.10) \quad A\eta(t) = g(x(t), By_t(t)) - \frac{d^2 x}{dt^2}$$

in the sense of  $Y^*$ .

In order to show the equation  $\eta(t) = F(x(t))$  one can use the monotonicity of  $F$ . So the following inequation holds

$$\begin{aligned} \langle A \circ F(Az) - A \circ F(Ay), z - y \rangle &= \langle F(Az) - F(Ay), Az - Ay \rangle = \\ &= \langle F(\tilde{x}) - F(x), \tilde{x} - x \rangle \geq 0 \end{aligned}$$

for any  $y, z \in Y$ ,  $Ay = x$  and  $Az = \tilde{x}$  by condition (i). Then one can write

$$0 \leq \langle F(x_m) - F(\tilde{x}), x_m - \tilde{x} \rangle = \langle F(Ay_m) - F(Az), Ay_m - Az \rangle =$$

take account here the equation (2.5)

$$(2.11) \quad \begin{aligned} &\langle F(Ay_m), Ay_m \rangle - \left\langle \frac{d^2}{dt^2} x_m - g(x_m, By_{mt}), z \right\rangle - \langle F(Az), Ay_m - Az \rangle = \\ &\langle F(x_m), x_m \rangle - \left\langle \frac{d^2}{dt^2} x_m - g(x_m, By_{mt}), z \right\rangle - \langle F(\tilde{x}), x_m - \tilde{x} \rangle. \end{aligned}$$

Here one can use the well-known inequation

$$\limsup \langle F(x_m), x_m \rangle \leq \langle \eta, x \rangle = \langle \eta, Ay \rangle = \langle A\eta, y \rangle.$$

Then passing to the limit in (2.11) with respect to  $m : m \nearrow \infty$  we obtain

$$0 \leq \langle A\eta, y \rangle - \left\langle \frac{d^2}{dt^2} x - g(x, By_t), z \right\rangle - \langle F(\tilde{x}), x - \tilde{x} \rangle =$$

$$\langle A\eta, y \rangle - \langle A\eta, z \rangle - \langle F(\tilde{x}), Ay - Az \rangle = \langle A\eta - A \circ F(\tilde{x}), y - z \rangle$$

by virtue of (2.10)

Consequently we obtain, that the equation  $A\eta(t) = A \circ F(x)$  holds since  $z$  is arbitrary element of  $Y$ .

Now for us is remained to show the obtained function  $x(t) = Ay(t)$  satisfy the initial conditions. Consider the following equation

$$\langle y_{mt}, Ay_m \rangle(t) = \int_0^t \left\langle \frac{d^2}{ds^2} Ay_m, y_m \right\rangle ds + \int_0^t \left\langle \frac{d}{ds} By_m, \frac{d}{ds} By_m \right\rangle ds + \langle y_{1m}, Ay_{0m} \rangle =$$

$$\int_0^t \left\langle \frac{d^2}{ds^2} y_m, Ay_m \right\rangle ds + \int_0^t \left\| \frac{d}{ds} By_m \right\|_H^2 ds + \langle y_{1m}, Ay_{0m} \rangle$$

for  $m = 1, 2, \dots$ , here  $x_m(t) = Ay_m(t)$ . Hence we get: the left side is bounded as far as all addings items in the right side are bounded by virtue of the obtained estimations. Therefore one can pass to limit with respect to  $m$  as here  $y_{mt}$  are continous with respect to  $t$  for any  $m$  then  $y_{mt}$  strongly converges to  $y_t$  and  $Ay_m$  weakly converges to  $Ay$  in  $H$ . It must be noted the equation

$$\lim_{m \rightarrow \infty} \int_0^t \left\| \frac{d}{ds} By_m \right\|_H^2 ds = \int_0^t \left\| \frac{d}{ds} By \right\|_H^2 ds$$

holds by virtue of the above reasonings that  $\{y_m(t)\}_{m=1}^\infty$  is a precompact subset in  $C^1(0, T; V)$ . Consequently the left side converges to the expression of such type, i.e. to  $\langle y_t, Ay \rangle(t)$ .

The obtained results shows that the following convergences are just:  $x_m(t) = Ay_m(t) \rightharpoonup Ay(t) = x(t)$  in  $X$ ,  $x_{mt}(t) = Ay_{mt} \rightharpoonup Ay_t = x_t(t)$  in  $V^*$ . From here follows, that the initial conditions are fulfilled in the sense of  $X$  and  $V^*$ , respectively.

Thus the following existence resut is proved.

**Theorem 1.** *Let spaces  $H, V, W, X, Y$  that are defined above satisfy all above mentioned conditions and condition (i)-(iii) are fulfilled then problem (1.1) - (1.2) is solvable in the space  $C^0(0, T; X) \cap C^1(0, T; V) \cap C^2(0, T; Y^*)$  for any  $x_0 \in V \cap [X^*, Y]_{\frac{1}{2}}$  and  $x_1 \in H$  in the sense of Definition 1.*

**Remark 1.** *This theorem shows that there exist a flow  $S(t)$  defined in  $V \times X$  and the solution of the problem (1.1) - (1.2) one can represent as  $x(t) = S(t) \circ (x_0, x_1)$ .*

## 3. BEHAVIOUR OF SOLUTIONS OF PROBLEM (1.1) - (1.2)

Here we consider problem under the following complementary conditions:

(iv) Let  $g(x, By_t) = 0$  and  $\|x\|_H^p(t) \leq c_0 \Phi(x(t))$  for some  $c_0 > 0$ .

We set a function  $E(t) = \|Bw\|_H^2(t)$  and consider this function on the solution of problem (1.1) - (1.2), then for  $E(t) = \|By\|_H^2(t)$  we have

$$(3.1) \quad E'(t) = 2 \langle By_t, By \rangle \leq \|By_t\|_2^2(t) + \|By\|_2^2(t),$$

where  $y = A^{-1}x$ . Here we will use equation (2.5). For this we lead the following equation

$$\frac{1}{2} \|By_s\|_H^2(s) + \Phi(x(s)) \Big|_0^t = 0$$

as  $g(x, By_t) = 0$ .<sup>1</sup> Hence

$$\frac{1}{2} \|By_s\|_H^2(t) + \Phi(x(t)) = \frac{1}{2} \|By_1\|_H^2(t) + \Phi(x_0)$$

and

$$\|By_t\|_H^2(t) = -2\Phi(x(t)) + \|By_1\|_H^2 + 2\Phi(x_0).$$

Granting this in (3.1) we get

$$E'(t) \leq E(t) - E^r(t) + \|By_1\|_H^2 + 2\Phi(x_0)$$

by virtue of the condition  $\Phi(x) \geq c_0 \|x\|_X^p$  and of the continuity of embedding  $X \subset H$ ,  $r = p/2$ .

So denoted by  $z(t) = E(t)$  we have the Cauchy problem for differential inequality

$$(3.2) \quad z'(t) \leq z(t) - cz^r(t) + C(x_0, x_1), \quad z(0) = \|By_0\|_H^2,$$

that we will investigate. Inequation (3.2) one can rewrite in the form

$$(z(t) + kC(x_0, x_1))' \leq z(t) + kC(x_0, x_1) - \delta [z(t) + kC(x_0, x_1)]^r,$$

where  $k > 1$  is a number and  $\delta = \delta(c, C, k, r) > 0$  is sufficiently small number.

Then solving this problem we get

$$z(t) + kC(x_0, x_1) \leq \left[ e^{(1-r)t} (z_0 + kC(x_0, x_1))^{1-r} + \delta (1 - e^{(1-r)t}) \right]^{\frac{1}{1-r}}$$

or

$$(3.3) \quad E(t) \leq \left[ e^{(1-r)t} \left( \|By_0\|_H^2 + kC(x_0, x_1) \right)^{1-r} + \delta (1 - e^{(1-r)t}) \right]^{\frac{1}{1-r}} - kC(x_0, x_1)$$

$$\|By\|_H^2(t) \leq \frac{e^t \left( \|By_0\|_H^2 + kC(x_0, x_1) \right)}{\left[ 1 + \delta \left( \|By_0\|_H^2 + kC(x_0, x_1) \right)^{r-1} (e^{(r-1)t} - 1) \right]^{\frac{1}{r-1}}} - kC(x_0, x_1).$$

here the right side is greater than zero, because  $\delta \leq \frac{k-1}{k^r C^r}$  and  $2r = p > 2$ .

Thus is proved the result

**Lemma 1.** *Under conditions (i), (ii), (iv) the function  $y(t)$ , defined by the solution of problem (2.1)-(2.2), for any  $t > 0$  is contained in ball  $B_l^{X \cap V}(0) \subset X \cap V$  depending from the initial values  $(x_0, x_1) \in (X \cap V) \times H$ , here  $l = l(x_0, x_1, p) > 0$ .*

<sup>1</sup>We would like to note that this equation shows the stability of the energy of the considered system in this case.

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